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Lyapunov Exponents and Localization in Randomly Layered Media

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A variety of problems involving disordered systems can be formulated mathematically in terms of products of random transfer matrices, including Ising spin systems, optical and continuum mechanical wave propagation, and lattice dynamical systems. The growth or decay of solutions to these problems is governed by the Lyapunov spectrum of the product of these matrices. For continuum mechanical or optical wave propagation, the transfer matrices arise from the application of boundary conditions at the discontinuities of the medium. Similar matrices arise in lattice-based systems when the equations of motion are solved recursively. For the disordered lattice mechanical system, on which we focus in this paper, the scattering effects of the heterogeneities on a propagating pulse can be characterized by the frequency-dependent localization length—effectively the ''skin depth'' for multiple-scattering attenuation. Thus there is a close connection in these transfer matrix-based systems between localization and the Lyapunov spectrum. For the one-dimensional lattice, the matrices are 2×2 and, assuming certain models of disorder, both Lyapunov exponents are nonzero and sum to zero. Thus all propagating solutions are either exponentially growing or decaying. For higher dimensions the situation is more complicated since there is then a spectrum of exponents, making the calculations more difficult, and it is less clear just how to relate the Lyapunov exponents to a single localization length. Further, unlike for the Schrödinger equation, the transfer matrices associated with the lattice mechanical system are not symplectic. We describe a robust numerical procedure for estimating the Lyapunov spectrum of products of random matrices and show application of the method to the propagation of waves on a lattice. In addition, we show how to estimate the uncertainties of these exponents.

BACKGROUND: WAVE PROPAGATION IN DISORDERED MEDIA AND LOCALIZATION

Broadly speaking, the effects of multiple scattering on a wave propagating in a disordered medium are:

- The path length is increased.
- The pulse is dispersed.
- The pulse is attenuated as energy is shifted from the direct arrival into the multiple-scattering coda.

These ideas are well understood and can be explained via perturbation theory for weakly disordered media. (For an overview of the mathematical results see [AKP+91].) In this paper we will restrict our attention to systems which are composed of a disordered or random collection of homogeneous layers. As an example, consider the model shown in Fig. 1. Here we have chosen a pseudo-random sequence (to be described later) of layer thicknesses and material property (elastic stiffnesses, dielectric constant, etc). We can scale these to reflect typical values for different applications. Figure 2 shows a 2D acoustic finite-difference wave propagation simulation for such a model; at a given depth, the plot shows the energy recorded at that depth as a function of time. The first energy seen at any given depth is the primary down-going pulse. Coherent events sloping downward to the right (such as the primary arrival) correspond to down-going waves, while events sloping downward to the left correspond to up-going waves. In a simulation such as this it is possible to follow the envelope of the pulse as it propagates into the medium. Figure 3 shows a log–linear plot of the amplitude of this pulse. Since the medium is nonattenuative, the decay shown in this figure is due to multiple scattering and geometrical spreading. Disordered layered systems such as this are common in a number of fields, including optics [GKST94], seismology [WSN90], quantum mechanics [KKT83], Ising spin systems [Fuc90], and others.

There is a close connection between multiple-scattering attenuation and Anderson localization [And58]. The Schrödinger equation for a lattice with random potential is closely related to the frequency-domain equations of motion for an elastic lattice with randomly varying spring constants and masses. It was shown by Ishii [Ish73] that in 1D any degree of uncorrelated random disorder is sufficient to cause exponential localization of the eigenfunctions of the quantum mechanical or elastic disordered system. (An eigenfunction ψ_n is exponentially localized if it decays exponentially at large distances from its maximum, $|\psi_n| < |\psi_0| \exp(-\alpha|n|)$, where α is a positive constant and the maximum of ψ has been taken to be at $n = 0$.) As we will describe in more detail below, for independent random disorder this exponential localization follows from a theorem of Furstenberg [Fur63] once the underlying equations have been rewritten in terms of transfer matrices.

For wave propagation the localization length can be

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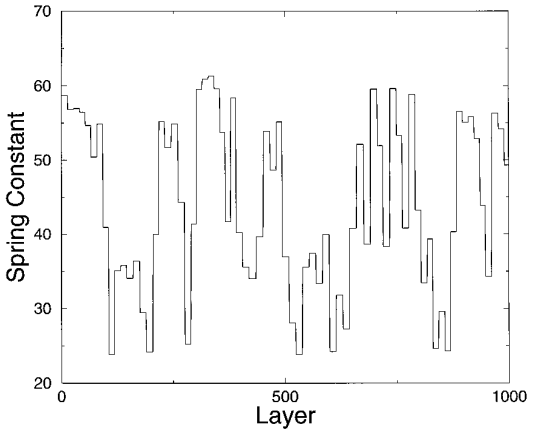

FIG. 1. A model of a randomly laminated medium obtained by choosing pseudo-random sequences of material property and layer thicknesses. These values can then be scaled to reflect typical material properties for a given application.

thought of as the skin depth associated with multiple-scattering attenuation. Consider a vertically incident plane wave in a layered medium. If the envelope of a plane wave decays exponentially with distance $e^{-z/l(f)}$, where f is the frequency, then $l(f)$ is the frequency-dependent localization length. Thus

$$l^{-1} = \lim_{L\to\infty}\left(-\frac{1}{L}\ln|T|\right), \tag{1}$$

where T is the transmission coefficient and L is the propagation distance. An approximation to the localization length for finite L is

$$l(f) \approx \frac{-L}{\ln|T|}. \tag{2}$$

The frequency-dependent localization length would be an extremely useful quantity to have in many applications involving wave propagation in highly heterogeneous media since it provides the means to scale the effects of complex microstructures. For example, given the exponential nature of the decay, it is natural to think of the multiple scattering as a relaxation mechanism. Hence there is a Q (quality factor) equal to $\pi fl/v(f)$, where v is the phase velocity [Fut62]. Then from Kramers–Krönig, the velocity dispersion is

$$\frac{1}{v(f)} - \frac{1}{v(\infty)} = \mathscr{H}\left(\frac{1}{2\pi fl}\right), \tag{3}$$

where $\mathscr{H}$ is the Hilbert transform, v_i are the layer velocities, $v(\infty) \equiv v_\infty = \langle v_i^{-1}\rangle^{-1}$ is the infinite frequency (ray theoretic) effective velocity, and the angle brackets denote averaging. (The zero-frequency effective velocity, the so-called Backus velocity, is $v_0 = \langle v_i^{-2}\rangle^{-1/2}$ [Bac62].)

A dispersion formula such as this tells us how to relate observations made at one length scale with those made at another. Sheng *et al.* [SZWP86a, SZWP86b] argue that for acoustic wave propagation, $l(f)$ has the form $c_1 + c_2/f^2$, in which case it is easy to show that the phase velocity dispersion formula is

$$v(f) = v_\infty\left(1 - \frac{v_\infty}{2\pi}\frac{\sqrt{c_2/c_1}}{c_1 f^2 + c_2}\right). \tag{4}$$

Rather than assuming a particular functional form for $l(f)$ we seek a direct method of calculating it. Returning to the finite-difference simulation shown above, it would appear that in principle one could perform a spectral analy-

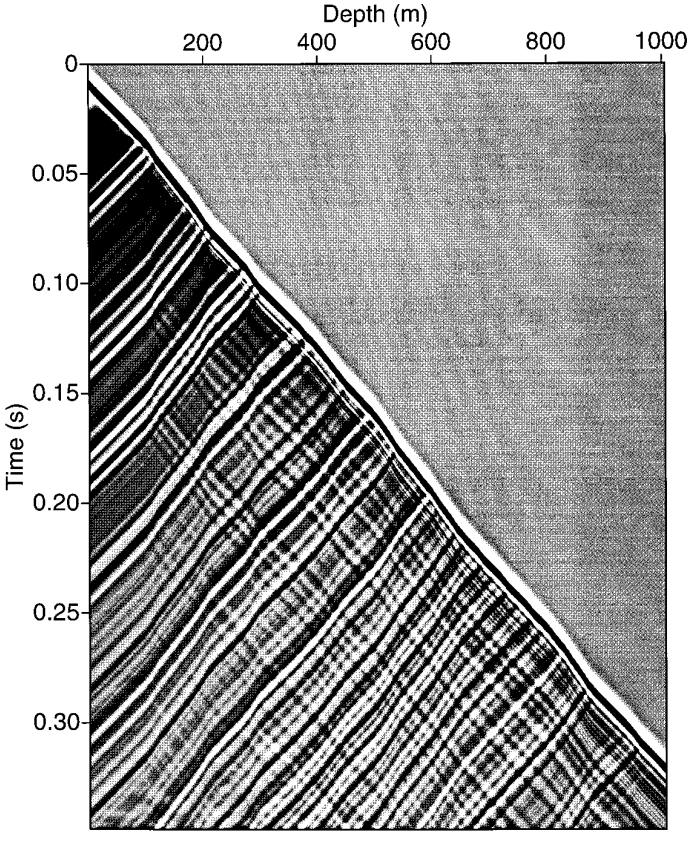

FIG. 2. We take the model shown in Fig. 1 and scale the spring constants and layer thicknesses to reflect values typical of the earth's near surface. Here is shown a 2D acoustic finite-difference simulation of a pulse propagating into such a medium. The pulse is generated by applying a band-limited point source (with a central frequency of 60 Hz) just below the upper surface. This figure shows the acoustic response at each layer as a function of time. The first event seen at each depth is the direct wave, with all the other events being up-going or down-going multiply scattered energy.

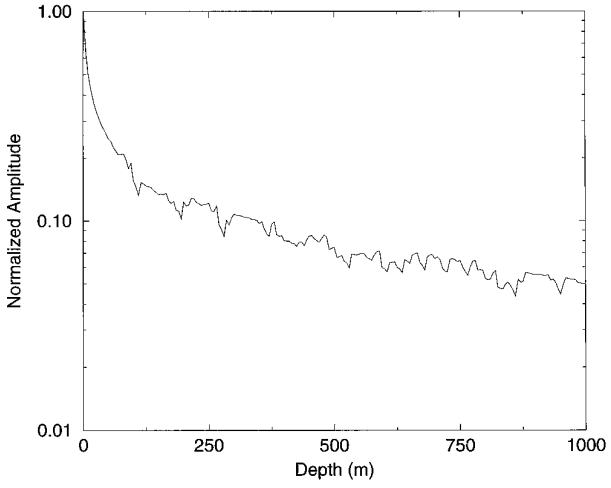

FIG. 3. Log–linear plot of the amplitude of the envelope of the downgoing pulse as a function of depth. Since the medium is perfectly elastic, this decay is due to multiple scattering and geometrical spreading.

sis of the observed decay of the pulse and from this estimate the frequency-dependent localization length. However, our goal is to develop robust numerical procedures for characterizing this multiple-scattering attenuation directly from *in situ* measurements of the material properties of the system, which we regard as a realization of some underlying stochastic process. We will achieve this by casting the wave propagation problem in terms of the products of random matrices. This is possible for 1D systems or quasi-1D systems such as the laminated medium described above, in which the wave propagation reduces to the application of transfer matrices. We will show later that the decay of energy in these quasi-1D systems, which we characterize in terms of the localization length, can be computed from the Lyapunov exponents (LEs) of the product of random transfer matrices. Further, we will describe a numerical algorithm based on the repeated application of the QR factorization for computing LEs, which allows us to compute just the positive LEs along with error estimates.

LATTICE DYNAMICAL SYSTEMS

The techniques we will present for the calculation of Lyapunov spectra can be applied to any problem that can be described in terms of the products of random matrices. Many examples of such problems are described in the books by Crisanti *et al.* [CPV93] and Mehta [Meh91]. Here, to illustrate the computational methods and to make the connection between the Lyapunov exponents and localization, we will focus attention on the lattice mechanical system. In this setting localization can be seen to arise from the tendency of lattice heterogeneities to convert propagating energy into localized fluctuations, which can themselves be visualized via the normal modes of the system. The localization length arises naturally as the reciprocal of one of the Lyapunov exponents of the product of transfer matrices for the lattice. There is a close connection between wave propagation on a lattice and wave propagation in a continuum, a connection that is discussed in detail in the book by Askar [Ask85].

EQUATIONS OF MOTION

Let us begin with the simplest case, waves propagating in a heterogeneous 1D lattice. Later, we will show how to generalize these results to higher dimensional systems.

There are N point masses m_l connected by $N-1$ Hooke's law springs of stiffness k_l. The end masses are fixed, as shown in Fig. 4, and the longitudinal displacement of the lth mass x_l is measured relative to its equilibrium position.

The kinetic energy of the system is $\frac{1}{2}\sum_{l=0}^{N} m_l \dot{x}_l^2$ and the potential energy (assuming only nearest-neighbor interactions) is $\frac{1}{2}\sum_{l=0}^{N} k_{l+1}\,(x_{l+1}-x_l)^2$. Thus the lattice equations of motion are

$$m_l \ddot{x}_l - k_{l+1}(x_{l+1}-x_l) - k_l(x_{l-1}-x_l) = 0. \tag{5}$$

Assuming a sinusoidal time dependence, the equation for the spatial part of the motion is

$$-m_l\omega^2 z_l = k_{l+1}z_{l+1} - (k_{l+1}+k_l)z_l + k_l z_{l-1}, \tag{6}$$

where $x_l \equiv z_l e^{i\omega t}$, or

$$(T+\omega^2 M)\mathbf{z} = 0, \tag{7}$$

where M is the mass matrix $M = \mathrm{diag}(m_l)$ and T is the tridiagonal matrix of spring constants: $T(l,l) = -(k_{l+1}+k_l)$, $T(l,l+1) = k_{l+1}$, and $T(l+1,l) = k_l$.

Free Oscillations of the 1D Lattice

For the ordered monoatomic 1D lattice $m_l = m$, $k_l = k$ for all l, so the equation for the free oscillations reduces to

$$(T+\omega^2 I)\mathbf{z} = 0. \tag{8}$$

Since the spring constants are all equal to k, T reduces to k times

$$\begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}. \tag{9}$$

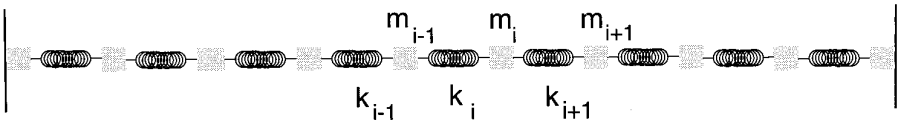

FIG. 4. A simple mechanical system exhibiting localization of energy. Masses m_i are connected via springs of stiffness k_i. The end masses are fixed. Longitudinal displacement of the ith mass x_i is measured relative to its equilibrium position.

The eigenvalues of this matrix are

$$\omega_l^2 = -2 + 2\cos\left(\frac{l\pi}{N}\right) \tag{10}$$

while the eigenvectors are

$$\mathbf{z}_l = \{\sin(l\pi/N), \sin(2l\pi/N), ..., \sin((N-1)l\pi/N)\}. \tag{11}$$

Another way of looking at the free oscillations of the lattice is to recast the eigenvalue problem in Eq. (6) as a recursion relation,

$$z_{l+1} = \left(2 - \frac{m\omega^2}{k}\right) z_l - z_{l-1}, \tag{12}$$

which can be rewritten as a one-step mapping

$$\begin{pmatrix} z_l \\ z_{l+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 - \dfrac{m\omega^2}{k} \end{pmatrix} \begin{pmatrix} z_{l-1} \\ z_l \end{pmatrix}. \tag{13}$$

Defining the matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}, \tag{14}$$

where $a = 1 - (m/2k)\omega^2$, it follows by induction that

$$\begin{pmatrix} z_l \\ z_{l+1} \end{pmatrix} = B^l \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \tag{15}$$

where B^l is the lth power of the matrix B.

The eigenvalues of B are $\lambda_{1,2} = a \pm \sqrt{a^2 - 1}$. If $|a| > 1$ ($\omega^2 > 4k/m$), then the z_l solutions are exponentially growing or decaying and cannot satisfy the zero-displacement boundary conditions. If $|a| < 1$ ($\omega^2 < 4k/m$), then the solutions are oscillatory and can be made to satisfy the boundary conditions. Taking $z_0 = 0$, then

$$\begin{pmatrix} z_{N-1} \\ z_N \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}^N \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16}$$

Obviously this equation cannot produce an eigenvector for arbitrary values of a. For example, if we are trying to compute normal modes whose displacement vanishes at both ends of the lattice, then only using a associated with an eigenvalue will result in z_N being zero.

LOCALIZED STATES ON LATTICES

For the homogeneous lattice all the eigenmodes are nonzero everywhere except at the nodal points. In that sense they have global extent. Suppose we disturb the regularity of the lattice by perturbing either a spring constant or a mass. Rayleigh's principle [Ray45, Vol. I, Sect. 88] says that if a single mass is reduced (or a spring constant increased) then all the frequencies are unchanged or increased, but not by more than the distance to the nearest unperturbed frequency. Similarly, if a single mass is increased (or a spring constant decreased) then all the frequencies are unchanged or decreased, but not by more than the distance to the nearest unperturbed frequency.

In spite of this apparent symmetry between the effects of increasing versus decreasing masses (or decreasing versus increasing spring constants), there is a profound difference in practice. Because the low-frequency limit is zero (at least for the monoatomic chain), decreasing a spring constant, and hence one or more frequencies, cannot result in new frequencies outside the band of frequencies allowed by the dispersion relation for the homogeneous lattice. On the other hand, increasing a spring constant can result in perturbing an eigenfrequency beyond the maximum allowed by the homogeneous dispersion relation ($2\sqrt{k/m}$ for the monoatomic chain). If this happens, the resulting eigenmode must be exponentially damped. One way to see that this is so is to observe that for complex wave numbers the sine in the dispersion relation becomes a sinh, which can match any frequency whatsoever. The result also follows from the discussion following Eq. (15).

This effect is illustrated in Fig. 5 which shows the two highest frequency eigenmodes for systems of point masses connected by 50 springs (spring constant equal to one) with, respectively, one and two perturbed spring constants.

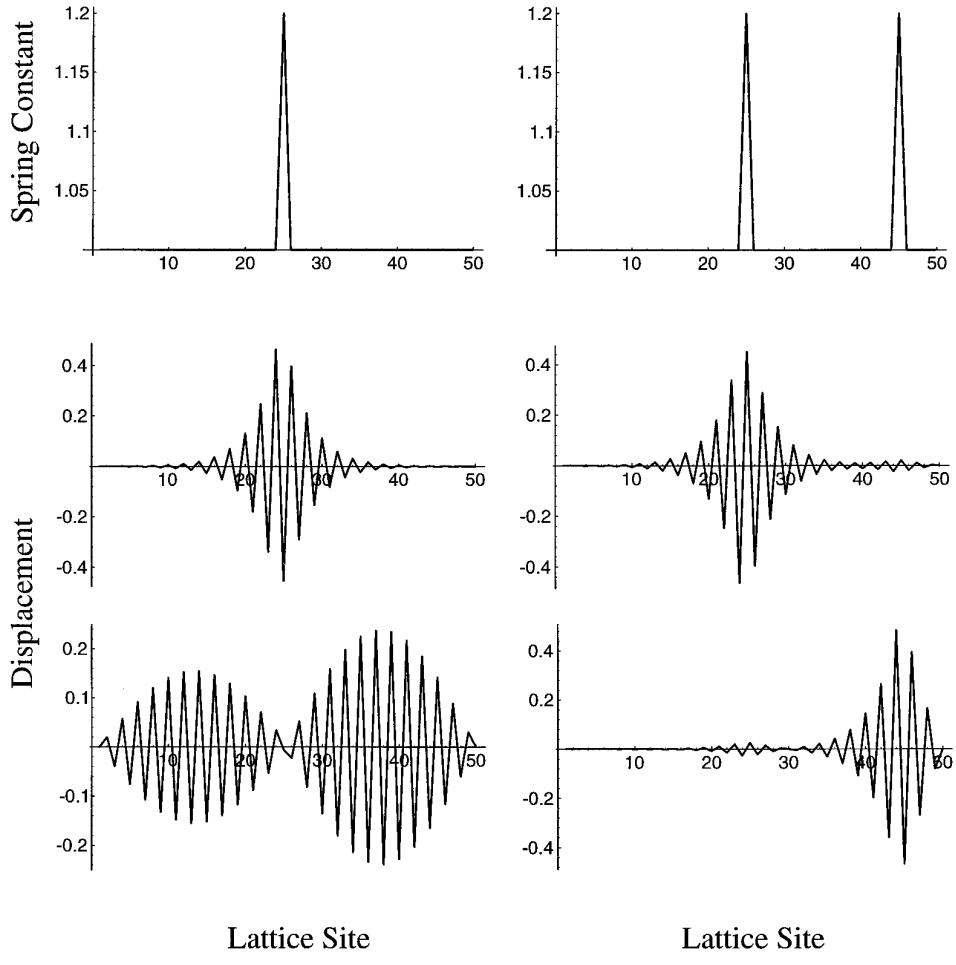

FIG. 5. As an example, consider a lattice of 50 unit masses connected by springs with unit spring constant. Perturb a single site by increasing its spring constant (or decreasing its mass). Rayleigh's principle says that this will result in increasing (or leaving unchanged) the frequencies. Increasing a single spring constant sufficiently results in a single eigenvalue being pushed out of the band of frequencies allowed by the homogeneous dispersion relation. This must therefore be associated with a localized mode. Perturbing two lattice sites in this case pushes two frequencies outside the allowed band, resulting in two localized modes.

In the first case, a single frequency is pushed into the forbidden band beyond $2\sqrt{k/m}$, resulting in a single localized eigenmode. Increasing two spring constants sufficiently results in two eigenfrequencies in the forbidden (or *impurity*) band and therefore two localized eigenmodes. (It is possible, by making a sufficiently large change in a single spring constant, to generate more than one localized mode centered on the impurity.) As a single spring constant is perturbed by ε, then the envelope of the highest frequency mode switches from convex to concave (i.e., becomes exponentially damped) at precisely that value of ε which pushes the first perturbed eigenfrequency into the impurity band.

The corresponding experiment, that of decreasing a single spring constant, does not result in a localized mode because the minimum frequency for the monoatomic chain is zero. For a chain with more than one type of ''atom'', the dispersion curve is multivalued, so we should expect to see localized modes associated with decreasing a spring constant since the higher frequency branches of the dispersion curve will have nonzero minimum frequency; pushing a frequency off the bottom of an upper branch would necessarily result in an exponentially damped eigenmode, just as pushing a frequency off the top of the acoustical branch does.

The physics of randomly disordered lattices began to be studied intensively in the 1940s and 1950s. A pioneering series of papers by Lifshitz [Lif43a, Lif43b, Lif44] was published in Russia during the Second World War. This phase of research culminated in the seminal paper by Anderson [And58]. Anderson's model was of lattices—regular or irregular—of electron spins (or other entities), each of whose energy was a random variable. Then, provided that the interparticle potential decayed sufficiently

fast as a function of distance (faster than r^{-3}), and that the disorder was stronger than some critical value, Anderson was able to show that the electron wave function was localized in space asymptotically with time. For a recent review of localization theory and experiment, see Kramer and MacKinnon [KM93].

Products of Random Matrices

The first systematic study of the randomly disordered 1D chain was made by Dyson [Dys52]. Dyson developed a method for calculating the distribution of eigenfrequencies in the $N \to \infty$ limit as a continued fraction. If the oscillators themselves are distributed according to an exponential law, Dyson's result is analytic. This work was extended by Schmidt [Sch57], who developed the recursive method of computing the eigenfunctions.

The eigenvector recursion formula extends readily to arbitrarily disordered chains. In this case the equations of motion are

$$z_{l+1} = \frac{k_{l+1} + k_l - m_l\omega^2}{k_{l+1}} z_l - \frac{k_l}{k_{l+1}} z_{l-1}, \tag{17}$$

which we can rewrite as

$$\begin{pmatrix} z_l \\ z_{l+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k_l}{k_{l+1}} & \dfrac{k_{l+1} + k_l - m_l\omega^2}{k_{l+1}} \end{pmatrix} \begin{pmatrix} z_{l-1} \\ z_l \end{pmatrix}. \tag{18}$$

Defining the matrix

$$B_l = \begin{pmatrix} 0 & 1 \\ -\dfrac{k_l}{k_{l+1}} & \dfrac{k_{l+1} + k_l - m_l\omega^2}{k_{l+1}} \end{pmatrix}, \tag{19}$$

it follows by induction that

$$\begin{pmatrix} z_l \\ z_{l+1} \end{pmatrix} = B_l \cdot B_{l-1} \cdots B_1 \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \equiv P_l \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{20}$$

In this way we have managed to reduce the study of wave propagation in disordered 1D media to application of the theory of products of random matrices (PRM). Later we will show how to generalize this approach to higher dimensions. We note that for constant spring constants and varying masses the coefficient matrix is symplectic, i.e., $J^T B_l J = B_l$ for all l, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

Also in this case, the determinant of each transfer matrix is 1 and hence the determinant of the product P_N is also 1. In general, for varying spring constants, the matrices B_l are not symplectic, nor are the determinants equal to 1. But since the determinant of the product P_N is the product of the determinants of the individual B_l, the determinant of P_N equals k_1/k_N. In the 1D numerical experiments described below, with pseudorandom k_i, we have always found there to be one positive and one negative LE.

Using the transfer matrix approach it is clear that the growth or decay of solutions to the equations of motion is governed by the exponential growth and decay rates of the product matrix P_N. Before we describe the interpretation of the Lyapunov spectrum vis-à-vis localization, we need to introduce a few basic facts about products or random matrices, Lyapunov exponents, and how we calculate them.

Asymptotic Properties of Products of Random Matrices

The first basic result is due to Furstenberg [Fur63], who showed that if the matrices are independent and nonsingular in a certain sense, the limit

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \ln \|P_N\| \tag{22}$$

exists almost surely, where here and throughout $\|\cdot\|$ denotes the Euclidean norm. Further, the maximum Lyapunov exponent λ_1 is a nonrandom quantity; in other words,

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \langle \ln \|P_N\| \rangle, \tag{23}$$

where the angle brackets refer to the average under the distribution associated with the B_l, which are assumed to be independent and identically distributed. The Lyapunov exponent measures the growth rate associated with *typical* vectors z,

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \ln \|P_N z\|, \tag{24}$$

although this does not preclude different growth rates for improbable choices of z.

These results were generalized by Virster [Vir79], who showed that if the matrices B_l were of the form $B_l = B(\xi_l)$, where ξ_l is a stationary ergodic Markov chain and B is a matrix function on the state space of the chain, then λ_1

exists and is positive almost surely. Virster's results, which are equivalent to Furstenberg's in the special case of independent matrices, thus apply to a rather large class of disordered systems including, for example, exponentially correlated Gaussian systems.

Calculation of the Lyapunov Spectrum

To determine the Lyapunov exponents of our linear mapping we will employ a method based upon performing a QR decomposition at each iteration. The discrete QR algorithm to be described below is the most widely used technique for approximating LEs. Other methods based upon performing a singular value decomposition are also possible (see Abarbanel *et al.* [ABK92] and Geist *et al.* [GPL90]). We consider the rates of growth and decay for the linear mapping

$$P_{N+1} = B_N P_N, \quad P_N \in \mathbb{R}^{n\times n}, N = 0, 1, ..., P_0 = I. \quad (25)$$

Lyapunov exponents are defined only for linear systems [Lya49, DRVV97]. For nonlinear systems the Lyapunov exponents with respect to a trajectory are defined using the linearization about this trajectory. In other words, the linear system in this case is the linear variational equation about the trajectory of the nonlinear system.

DEFINITION [Ose68]. Let $P_{N+1} = B_N \cdots B_0$, $N = 1, ...,$ be a fundamental solution of (25) (with $(P_0)^T P_0 = I$). Then, the following symmetric positive definite matrix exists,

$$\Lambda = \lim_{N\to\infty} (P_N{}^T P_N)^{1/2N},$$

the logarithms of the eigenvalues of which are called *Lyapunov exponents*, and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The λ_i's do not depend on the initial condition matrix P^0 almost surely.

The theorem of Oseledec [Ose68] leads to an equivalent characterization of LEs. Let $\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(p)}$ be the LEs of (25) *not repeated by multiplicity*, where p denotes the number of distinct exponents. Let $E^{(i)}$ be the invariant subspace of $\mathbb{R}^n$ corresponding to the eigenvalues of Λ whose logarithm is less than or equal to $\lambda^{(i)}$, so that $\mathbb{R}^n = E^{(1)} \supset E^{(2)} \supset \cdots \supset E^{(p)}$. Let $\mathbf{p}_k \in E^{(k)} \backslash E^{(k+1)}$, for $k = 1, ..., p$, where $E^{(p+1)}$ is the empty set. Then one has

$$\lambda^{(k)} = \limsup_{N\to\infty} \frac{1}{N} \log \|P_N \mathbf{p}_k\|.$$

For our n-dimensional problem the n Lyapunov exponents $\{\lambda_k\}_{k=1}^n$ consist of the distinct values $\lambda^{(1)}$, $\lambda^{(2)}$, ..., $\lambda^{(p)}$.

The system (25) is called *regular* [Lya49] if

$$\sum_{k=1}^{n} \lambda_k = \lim_{N\to\infty} \frac{1}{N} \log(|\det(B_{N-1} \cdots B_0)|) = \lim_{N\to\infty} \frac{1}{N} \log(|\det(P_N)|). \quad (26)$$

If (25) is regular and upper triangular, one has [Lya49]

$$\lambda_k = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \log|(B_j)_{kk}|, \quad k = 1, ..., p.$$

When the B_j are not upper triangular, we will successively compute QR factorizations of the transfer matrices. Given orthogonal P_0: $P_0^T P_0 = I$, let $Q_0 = P_0$. Set $Z_{N+1} = B_N Q_N$, $N = 0, 1, ...,$ and then decompose $Z_{N+1} = Q_{N+1} R_{N+1}$, where R_{N+1} is upper triangular with positive diagonal entries. Since we obtain $Q_{N+1} R_{N+1} = B_N Q_N$, then $Q_{N+1}^T B_N Q_N = R_{N+1}$ is upper triangular. Thus, we can obtain the LEs as

$$\lambda_k = \lim_{N\to\infty} \frac{1}{N} \log((R_N)_{kk} \cdots (R_1)_{kk}) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \log((R_j)_{kk}). \quad (27)$$

Following the work in [DRVV97, DVV95] we will estimate the error in our calculation by fitting the error as $K/N + \varepsilon$. If $\lambda_k(N)$ is the kth finite time Lyapunov exponent at iterate N, then simple linear regression gives us the following values for ε and K,

$$K = \frac{\sum_{n=1}^{N} \frac{\lambda_k(n)}{n} - \frac{1}{N}\left(\sum_{n=1}^{N} \frac{1}{n}\right)\left(\sum_{n=1}^{N} \lambda_k(n)\right)}{\sum_{n=1}^{N} \frac{1}{n^2} - \frac{1}{N}\left(\sum_{n=1}^{N} \frac{1}{n}\right)^2}$$

and

$$\varepsilon = \frac{1}{N}\left(\sum_{n=1}^{N} \lambda_k(n) - K \sum_{n=1}^{N} \frac{1}{n}\right) - \lambda_k^*,$$

where λ_k^* is the exact Lyapunov exponent. Since we do not know the exact Lyapunov exponent, we use the best available approximation, $\lambda_k(N)$, instead of λ_k^*.

To recap, we have described a technique for estimating the exponential growth or decay of products of random transfer matrices by computing the Lyapunov spectrum of this matrix product via a discrete QR method. Error estimates follow from linear regression of the finite time exponents.

Definition of the Localization Length

The localization length is often defined in terms of the exponential decay of eigenfunctions [KM93]. But since the Lyapunov exponents control the exponential growth/decay of solutions of the differential equation, it seems reasonable to associate the localization length with the reciprocal of one of the Lyapunov exponents (LEs). In 1D if $k_1 = k_N$ then the transfer matrices are unimodular and the two LEs are of equal magnitude and opposite sign. The positive one describes exponential growth in forward time (or exponential decay in reverse time), and vice versa for the negative exponent. In this case it is conventional to identify the localization length with the reciprocal of the positive LE [CPV93].

This identification is well supported by numerical evidence (cf. [CPV93, KM93], and the simulations below). However, in higher dimensions the situation is not so straightforward. As we will see presently, the transfer matrices for lattices in higher dimensions are no longer 2×2, in which case the dominant rate of growth/decay of solutions will be controlled by the smallest (in absolute value) positive/negative exponents. But we do not know *a priori* how many of the LEs are nonzero for arbitrary transfer matrices; nor can we be sure that the LEs come in pairs (equal in magnitude but opposite sign), as would be the case if the product matrix P_N were symplectic [Ott93, LVV97]. Thus there would seem to be several reasonable definitions of a localization length. Suppose that there are $2N_2$ exponents λ_i. Then we could reasonably define the localization length as the reciprocal of the magnitude of any of the following exponents: the smallest positive ($\equiv\lambda_{\text{pos}}$), the largest negative ($\equiv\lambda_{\text{neg}}$), λ_{N_2}, or λ_{N_2+1}. Later we will show that in our 2D lattice simulations these four definitions give nearly identical results. But for now we will define the localization length to be the reciprocal of λ_{N_2}.

Simulations—One Dimension

To do the simulations we generate pseudo-random sequences of spring constants and layer thicknesses. These are chosen in an uncorrelated fashion from uniform distributions and then correlated by applying a running average of a given length.

Each layer is described by its thickness and spring constant and the medium is described by a sequence of thicknesses and the corresponding spring constants in each layer. Thus, the statistical properties of our simulated medium are described in terms of maximum and minimum values and correlation lengths for both the thicknesses of the layers and the spring constants.

If the layer thicknesses were all equal to 1, say, then the maximum frequency of propagation on the lattice would correspond to a wavelength of 1. By putting groups of homogeneous spring constants between the "scatterers" we can simulate wavelengths smaller than the distance between the scatterers.

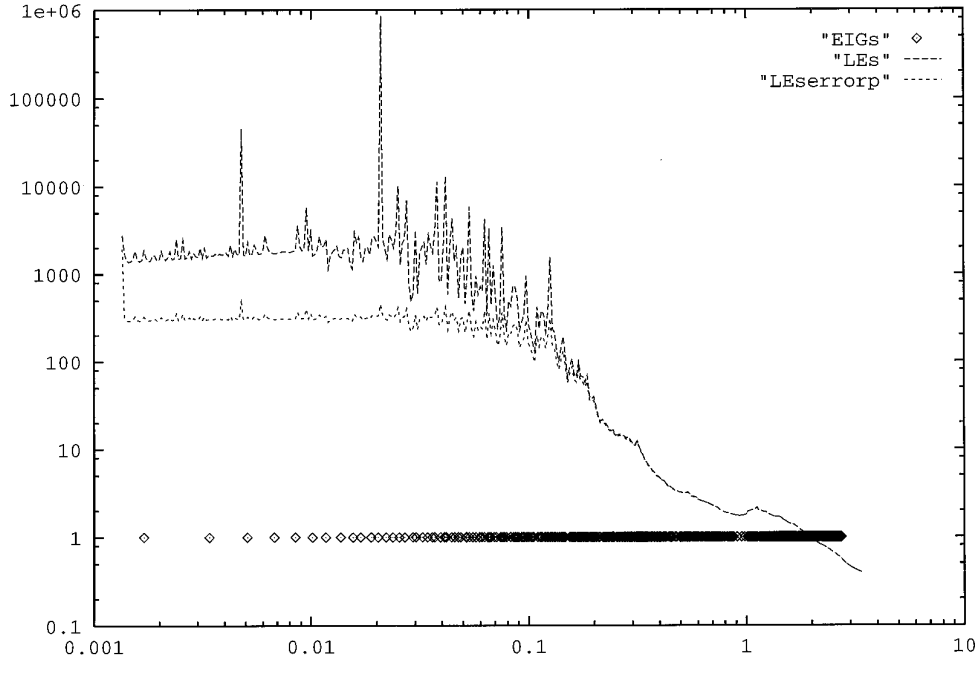

FIG. 6. $1/\lambda_1$ vs $\omega/2\pi$, where λ_1 is the positive Lyapunov exponent, spring correlation length = 1, thickness correlation length = 1, maximum thickness = 1, and minimum thickness = 1.

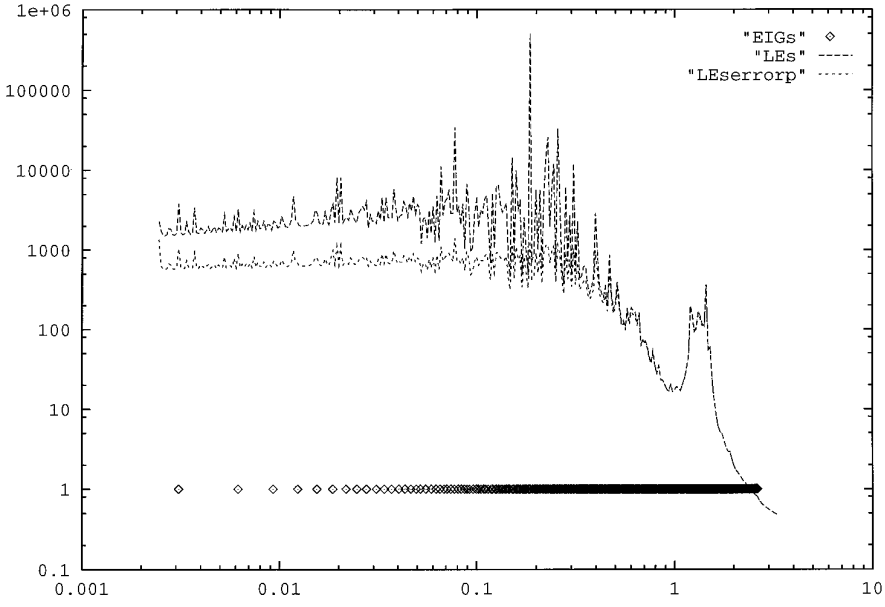

FIG. 7. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 2, thickness correlation length = 1, maximum thickness = 1, and minimum thickness = 1.

Once the medium is determined, the task is to compute the positive Lyapunov exponents. The Lyapunov exponents are defined as limits, although our medium will have a finite thickness. To obtain better approximations one could average over several starting vectors or, more easily, simply consider the medium as periodic and average over several periods.

In Figs. 6–12 we illustrate some of the numerical results we have obtained. All plots are in log–log scale and plot the frequency $\omega/(2\pi)$ on the horizontal axis against the computed localization length on the vertical axis, where the localization length is one over the positive LE. It is the frequency dependence of the localization length that gives insight into the transport of energy in the system, so it is this aspect that we focus on in these figures.

We use the computed upper bound for the Lyapunov exponent to compute a lower bound on the localization length; this is denoted by ''LEserrorp'' in the figures. The

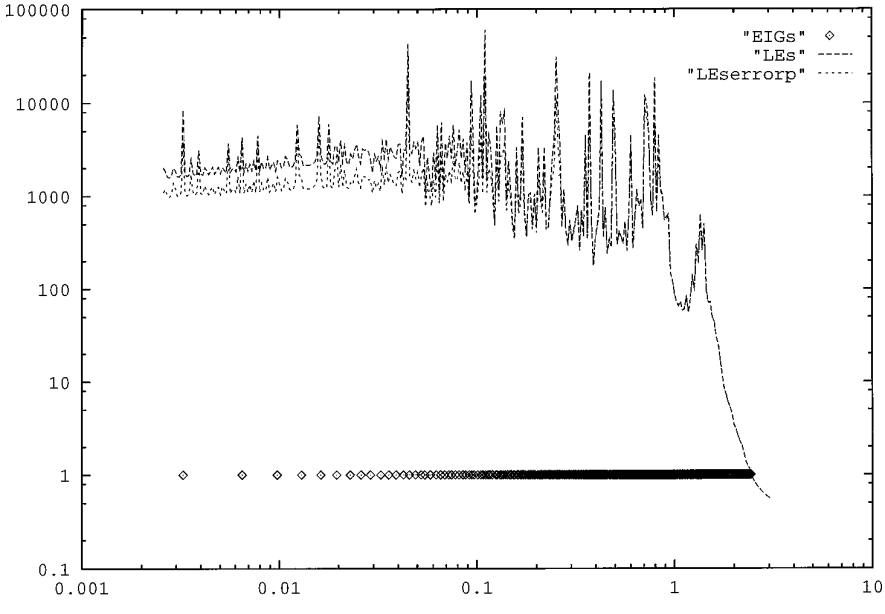

FIG. 8. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 4, thickness correlation length = 1, maximum thickness = 1, and minimum thickness = 1.

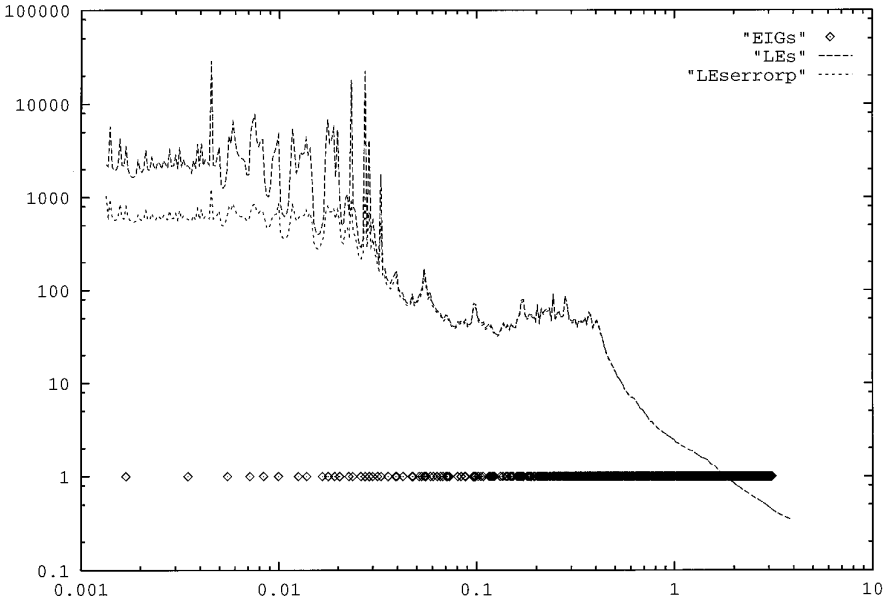

FIG. 9. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 1, thickness correlation length = 1, maximum thickness = 10, and minimum thickness = 5.

simulations are for media with a total thickness of 1000 and the Lyapunov exponent was obtained by averaging over 10 ''periods.'' In all of the simulations the spring constants had a maximum value of 100 and a minimum value of 1. The correlation length for the spring constants and the thicknesses was allowed to vary, as were the maximum and minimum thickness of the layers.

The maximum localization length computable in this way is the length of the lattice, in this case 1000. Thus, values of $1/\lambda_1$ greater than 1000 correspond to weak attenuation. On the other hand, the minimum meaningful localization length is the lattice spacing. We see that at sufficiently high frequencies the waves are localized. However, as the medium becomes more and more smooth, the range of frequencies that are localized goes to zero.

Figures 6–8 illustrate the lack of monotonicity that occurs for layers of unit thickness and small correlation lengths for the spring constants. All figures provide

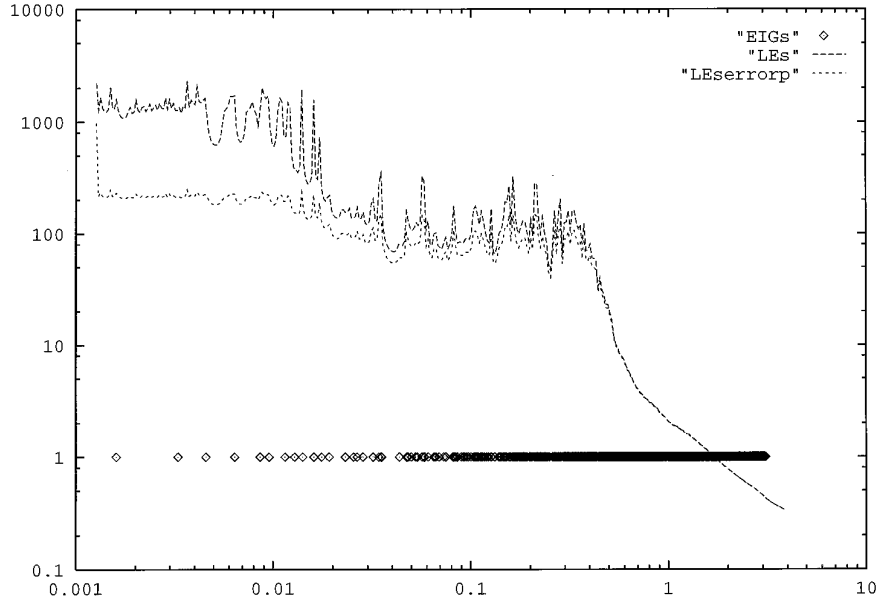

FIG. 10. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 1, thickness correlation length = 1, maximum thickness = 20, and minimum thickness = 10.

comparisons of the localization length and the spectrum of the tridiagonal matrix $-T$ (see Eqs. (6) and (7)). The local minima in the localization length appear to occur where there are gaps in the spectrum of $-T$. (Since the frequency-domain equations of motion involve ω^2 we are actually showing the square roots of the eigenvalues of $-T$.)

Next we show results for media composed of random "layers" of varying thickness. In other words, to make a layer of thickness n we put n springs of a given spring constant together. In Figs. 9 and 10 for thickness between 5 and 10, and 10 and 20, respectively, the localization length $l(f)$ is basically monotone, with regions of different behavior of $l(f)$ depending on whether l is large or small compared to the layer thickness.

For Figs. 11 and 12 we have thicknesses between 10 and 20. Also, the spring constant correlation lengths (thickness correlation lengths) are 1 and 4 (4 and 16), respectively. As the thicknesses and/or correlation lengths increase the localization length approaches that of a uniform medium, as expected.

The model used for the calculations in Fig. 12, which show a localization length greater than or equal to 1000 for almost all frequencies, is the same as that in Fig. 1. However, the results of the finite-difference calculation in Figs. 2 and 3 are for a 2D simulation, corresponding to the propagation of a point source in a layered 2D medium. For comparison with the 1D LE results we also carried out the 1D time-domain finite-difference calculation, corresponding to a vertically incident plane wave rather than a point source. The results are kinematically identical to Fig. 2, but amplitudes will be different due to the geometrical spreading. Figure 13 shows the corresponding 1D decay with distance of the time-domain pulse computed via acoustic finite differences. This linear–log plot is well fit by a straight line whose slope corresponds to a localization length of 1000. This is consistent with the theory since in 1D there is only one positive LE and so the decay must be a pure exponential; it also agrees quantitatively with the direct LE calculation. On the other hand, in the 2D finite difference results shown in Fig. 3, there are clearly two different length scales involved. Lagendijk *et al.* [LVAVDM86] argue that for strong scattering the first exponential decay length is associated with the scattering mean free path, while the second is the characteristic length associated with absorption. In the next section we will generalize our approach to allow for 2D lattices. There we will see that the corresponding 2D lattice results (Fig. 16) are indeed consistent with the time-domain finite difference.

Simulations—Two Dimensions

The extension of our approach to higher dimensional lattices presents some challenges. The theory of the products of random matrices (PRM) still applies to some extent, although with more complicated matrices. In two dimensions an approach similar to that considered above may be employed. Consider again Eq. (7) where now T represents the two-dimensional analog of the discrete Laplacian operator with random spring coefficients. To cast this problem in terms of propagator matrices some assumptions

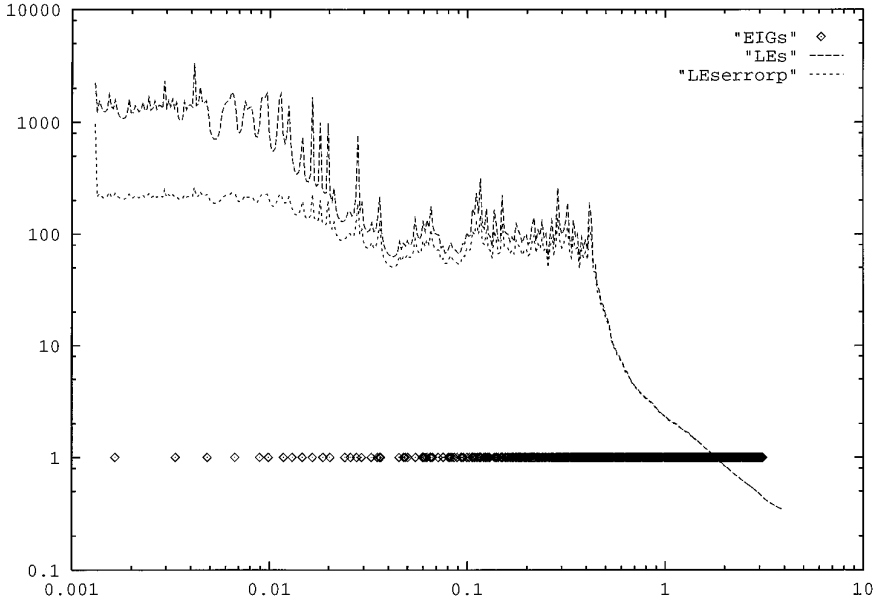

FIG. 11. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 1, thickness correlation length = 4, maximum thickness = 20, and minimum thickness = 10.

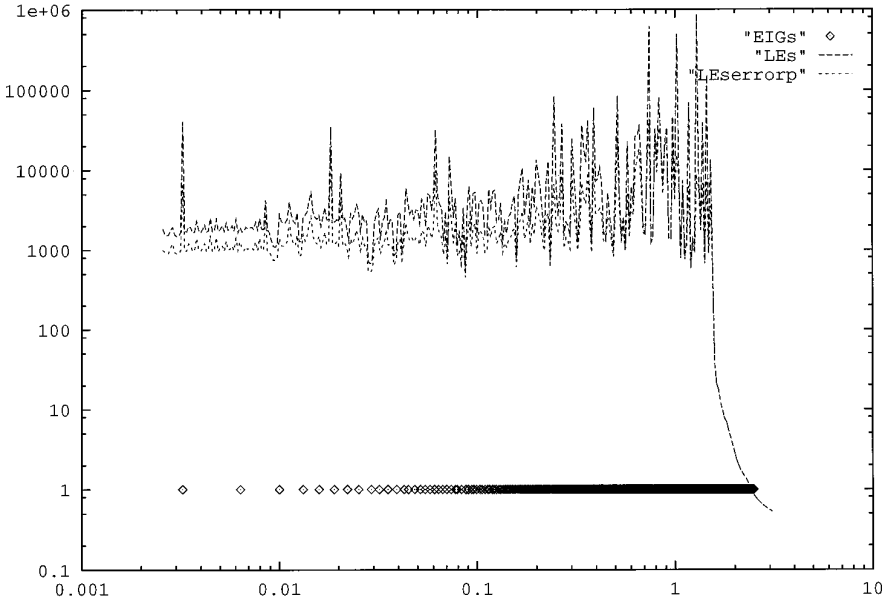

FIG. 12. $1/\lambda_1$ vs $\omega/2\pi$, spring correlation length = 4, thickness correlation length = 4, maximum thickness = 20, and minimum thickness = 10.

must be made. Here, for $z = \{z_{i,j}\}_{(i,j)=(1,1)}^{(N_1,N_2)}$ we impose ''corkscrew'' boundary conditions in one direction. (See [Fuc90] and [CMPVV96] for more details on this type of boundary condition.) The corkscrew boundary conditions allow us to linearly order the variables as

$$z_{1,1}, ..., z_{1,N_2}, z_{2,1}, ..., z_{2,N_2}, ...,$$

and write the evolution as a mapping whose Lyapunov exponents may be calculated. On the other hand, periodic boundary conditions may also be employed and this has been done for the Schrödinger equation (corresponding to constant spring constants and random masses) in Crisanti *et al.* [CPV93]. For the case of periodic boundary conditions, random masses, and constant spring constants,

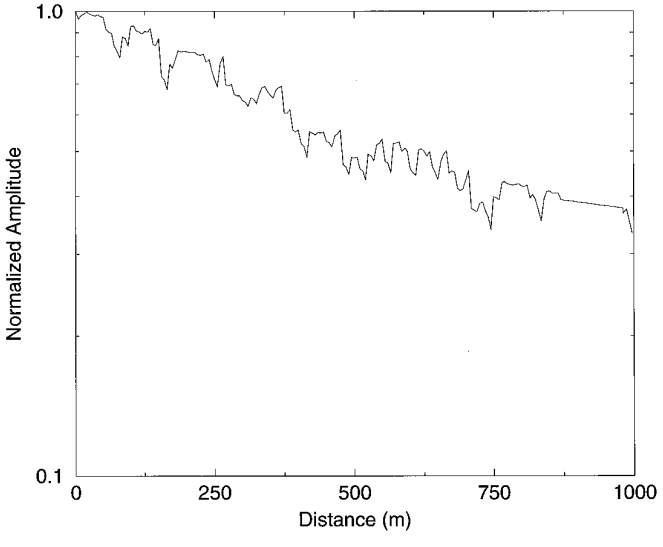

FIG. 13. Long–linear plot of the amplitude of the envelope of the down-going plane wave as a function of depth in a purely 1D medium shown in Fig. 1 computed via time-domain finite difference. The figure can be interpreted as saying that the localization length (averaged over all frequencies) is around 1000 for this model. This can be compared with the corresponding lattice result in Fig. 12, which shows a localization length of 1000 or more for nearly all frequencies.

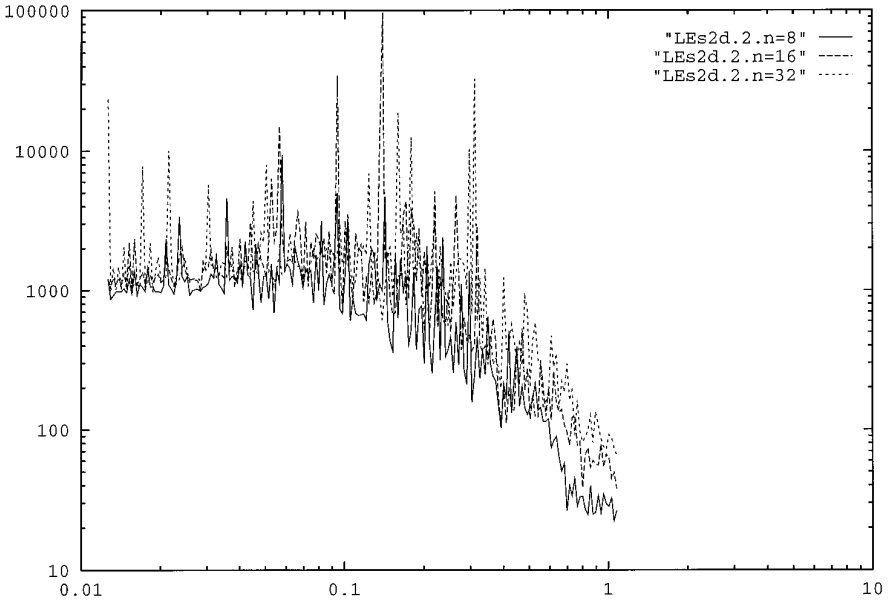

FIG. 14. $1/\lambda_{N_2}$ vs $\omega/2\pi$, spring correlation length = 2, thickness correlation length = 1, maximum thickness = 1, and minimum thickness = 1.

the matrices that are obtained are symplectic, in contrast with the matrices obtained below.

Let

$$z^{(1)} = (z_{1,1}, ..., z_{1,N_2}, z_{2,1}, ..., z_{2,N_2})^T,$$
$$z^{(2)} = (z_{1,2}, ..., z_{1,N_2}, z_{2,1}, ..., z_{2,N_2}, z_{3,1})^T,$$

etc.; then for $n = i + (j-1)N_2$, $1 \le i \le N_1$, $1 \le j \le N_2$, we have

$$z^{(n+1)} = A_n z^{(n)},$$

where

$$A_n = \begin{pmatrix} \mathbf{0} & I \\ -a_n & 0, ..., 0, -b_n, c_n, -d_n, 0, ... 0 \end{pmatrix}.$$

The coefficients of the matrix A_n are given in terms of the frequency and the spring constants as

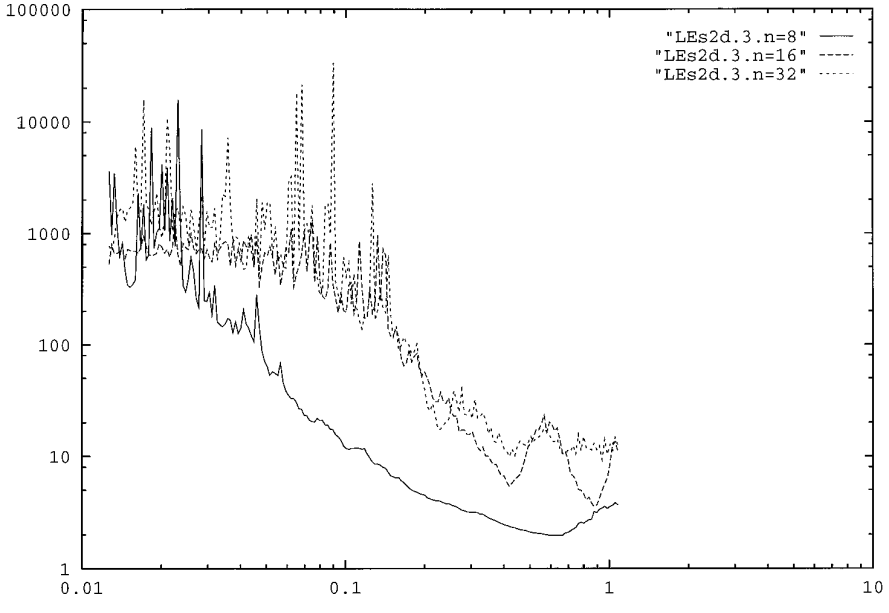

FIG. 15. $1/\lambda_{N_2}$ vs $\omega/2\pi$, spring correlation length = 4, thickness correlation length = 1, maximum thickness = 1, and minimum thickness = 1.

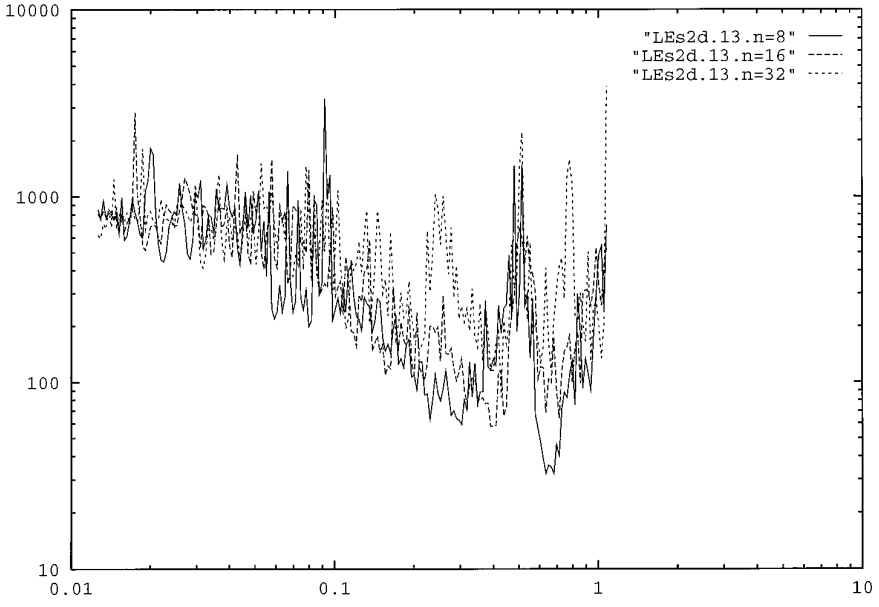

FIG. 16. $1/\lambda_{N_2}$ vs $\omega/2\pi$, spring correlation length = 4, thickness correlation length = 4, maximum thickness = 20, and minimum thickness = 10.

$$
\begin{aligned}
a_n &= k_n/k_{2_{N_2+n}},\\
b_n &= k_{N_2+n-1}/k_{2N_2+n},\\
c_n &= (k_n + k_{N_2+n-1} + k_{N_2+n+1} + k_{2N_2+n} - \omega^2)/k_{2N_2+n},\\
d_n &= k_{N_2+n+1}/k_{2N_2+n}
\end{aligned}
\tag{28}
$$

modulo the boundary conditions. Notice that for each of the matrices A_n the trace is zero and the determinant is $\pm a_n$.

In Figs. 14, 15, and 16 we illustrate the results of numerical experiments obtained for various values of the width N_2 computed over five periods. Note that the Lyapunov exponents are ordered according to

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{N_2} \geq \cdots \lambda_{2N_2}.$$

The pseudo-random models used in Figs. 14, 15, and 16 have the same statistical properties as those in Figs. 7, 8, and 12, respectively.

Figure 16 was computed for the model shown in Fig. 1 and therefore affords a direct comparison with the 1D LE calculation in Fig. 12 and the time-domain finite-difference results in Figs. 3 (2D) and 13 (1D). In 2D, as we would have expected from the finite-difference results, we see localization lengths less than 1000 for nearly all frequencies. In addition we see clear evidence of nonmonotonicity of the frequency dependence of the localization length.

To decrease the computation time we compute only the first N_2 exponents and not all $2N_2$ exponents (see [DVV95]). Note in Figs. 14 and 15 that it appears that the localization lengths are converging as the width N_2 is being increased.

That the Lyapunov exponents are nonrandom in the quasi-1D case follows under reasonable assumptions from the work of Oseledec [Ose68]. However, in the quasi-1D case there is no guarantee that the first N_2 finite time Lyapunov exponents are positive even if $\det(P_N) \geq 1$. As a consequence there appear to be several alternative definitions of the localization length. Obvious choices would be to use the absolute value of the reciprocal of one of the following LEs: the smallest positive ($\equiv\lambda_{\text{pos}}$), the largest negative ($\equiv\lambda_{\text{neg}}$), λ_{N_2}, or λ_{N_2+1}. If the product matrix P_N were symplectic and if all the LEs were nonzero, then these four definitions would be identical. In our calculations we have defined the localization length to be $1/\lambda_{N_2}$. However, in Fig. 17, we show all four definitions plotted simultaneously. The consistency of these different estimates is a good indication of how robust the numerical procedure is.

CONCLUSIONS

A variety of problems involving disordered systems can be formulated mathematically in terms of products of random transfer matrices. In such cases, the growth or decay of solutions is governed by the Lyapunov spectrum of the matrix product. We have presented an algorithm for computing the finite time Lyapunov exponents (along with error estimates) for such problems and applied the algorithm to the study of layered lattices in 1D and 2D. On

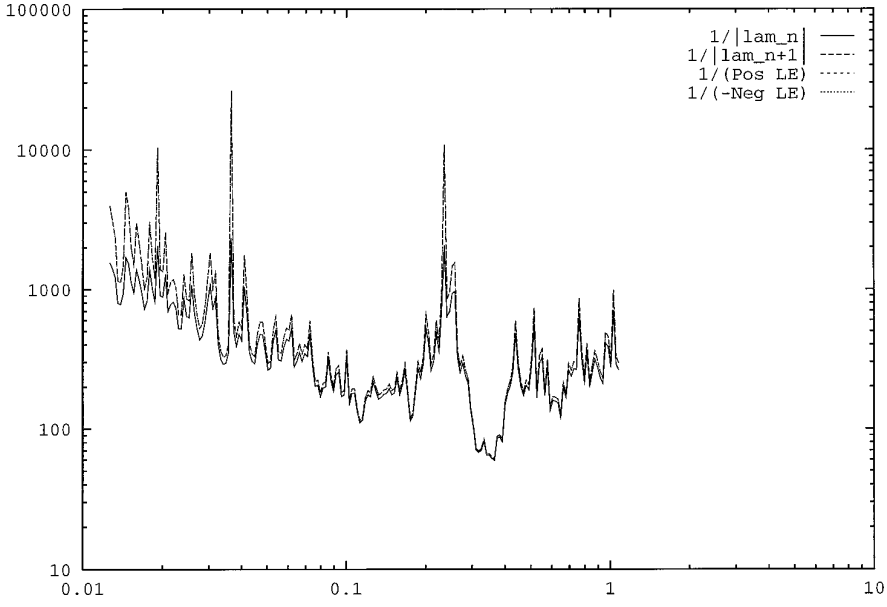

FIG. 17. The localization length as a function of frequency computed using four different definitions described in the text corresponding to the $n = 8$ case of Fig. 16. The consistency of these different estimates is an indication of the numerical robustness of the procedure.

the lattice, waves scattering from random heterogeneities may excite vibrational modes that are localized about the heterogeneities. This causes propagating energy to be converted into localized fluctuations and is manifested in the dispersion and attenuation of the waves. With a sufficiently strong random distribution of heterogeneities, it is possible for a propagating wave to become trapped altogether. The frequency-dependent length scale over which this trapping occurs—the localization length—is the fundamental quantity characterizing the phenomenon. The Lyapunov exponent calculation gives us a robust numerical procedure for computing the frequency-dependent details of localization directly from the properties of the medium. We have applied this method to pseudo-random lattices in 1D and 2D. Generally speaking, the smoother the medium, the longer the localization length. The fine details are more complicated however. For example, we see clear evidence of nonmonotonic behavior of the frequency-dependent localization length as the lattice correlation length is varied. A detailed explanation of this rich behavior is the subject of current research. Further, there are a number of theoretical challenges: while Virster's theorem tells us that for Markovian transfer matrices the largest LE is positive, in 2D and higher, we need to know when we can expect the first N_2 exponents to be positive. It would also be very useful to have a generalization of this theorem to non-Markovian matrices.

ACKNOWLEDGMENTS

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